

# A note on the Thom isomorphism in geometric (co)homology

Martin Jakob

February 1, 2008

## Abstract

Using geometric homology and cohomology we give a simple conceptual proof of the Thom isomorphism theorem.

## 1 Introduction

In the articles [4] and [5] we constructed homology functors in a geometric way. We showed that these functors which are defined on pairs of spaces satisfy the Eilenberg–Steenrod axioms for a generalized homology theory.

Let  $h^*$  be a multiplicative cohomology theory (for example singular cohomology with coefficients in a ring, or  $K$ –theory). In this setting, the associated homology groups  $h_q(X)$  of a space  $X$  are defined by means of of triples  $(M, x, f)$  (so-called *cycles*), where  $M$  is a manifold,  $x \in h^*(M)$  is a cohomology class of  $M$  and  $f : M \rightarrow X$  is a continuous map.

An equivalence relation on the set of cycles must be imposed. It includes bordism and a procedure called “vector bundle modification” to shift the degree of  $x$  and the dimension of  $M$  without leaving the class of  $(M, x, f)$ .

In the case of  $K$ –theory this construction goes back to some work of Paul Baum, cf. for example [1].

On the category of differentiable manifolds and smooth maps there is also a bivariant version of this construction (cf. [3]). For an introductory text to bivariant theories see [2].

These geometric approaches to homology as opposed to the usual spectral methods of stable homotopy allow surprisingly simple proofs of properties requiring orientations (say of manifolds or maps). In the usual stable homotopy setting often extra considerations are needed. An illustration of this point is the proof of the Poincaré duality theorem.

Recently, the question has been raised whether in the geometric (co)homology setting there is a simple proof of the Thom isomorphism theorem. That is in fact the case and in this little note we shall prove:

**Theorem 1.1** *Let  $h^*$  be a multiplicative cohomology theory and let  $\pi : E \rightarrow X$  be a smooth  $h^*$ -oriented vector bundle of rank  $n$ . Then the geometric Thom class*

$$t_E = [X, 1_X, \sigma_0] \in h^n(DE, SE)$$

*induces isomorphisms*

$$h^q(X) \rightarrow h^{q+n}(DE, SE), \quad x \mapsto t_E \cdot \pi^*(x)$$

*and*

$$h^{q+n}(DE, SE) \rightarrow h^q(X), \quad y \mapsto \pi_!(y)$$

*which are inverse to each other.*

Here  $DE$ , resp.  $SE$  is the total space of the unit ball, resp. unit sphere bundle associated to some metric on  $E$ , and  $\sigma_0$  denotes the zero section of the ball bundle. Further, the map  $\pi_!$  sends the geometric class  $[P, x, g] \in h^{q+n}$  onto the class  $[P, x, \pi g] \in h^q(X)$ .

To be precise, in geometric cohomology  $DE$  should be a manifold without boundary. So one has to put  $DE = \text{open ball bundle in } E \text{ of a fixed radius } > 1$ .

## 2 The proof of the Thom isomorphism theorem

Let us observe first that  $\sigma_0$  in the cycle  $(X, 1_x, \sigma_0)$  avoids the sphere bundle  $SE$  and thus describes in fact a geometric cohomology class of the pair  $(DE, SE)$ .

Let  $[M, x, f] \in h^q(X)$ . Now  $t_E \cdot \pi^*([M, x, f]) \in h^{q+n}(DE, SE)$  is represented by the composition of pull backs

$$\begin{array}{ccc} (M \times_X DE) \times_{DE} X & \xrightarrow{f''} & X \\ \downarrow & & \downarrow \\ M \times_X DE & \xrightarrow{f'} & DE \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X. \end{array}$$

Then

$$((M \times_X DE) \times_{DE} X, x \times 1 \times 1, f'')$$

represents  $\pi_!(t_E \cdot \pi^*([M, x, f]))$ . The equivalence to  $(M, x, f)$  is given by the isomorphism

$$(M \times_X DE) \times_{DE} X \longrightarrow M, \quad ((m, v), x) \mapsto m.$$

Therefore we have shown

$$\pi_!(t_E \cdot \pi^*(\dots)) = id_{h^*(X)}.$$

Let  $(P, x, g)$  be a cycle representing a cohomology class  $y$  of  $h^*(DE, SE)$ . Then  $t_E \cdot \pi^*(\pi_!(y))$  is represented by the pull back diagram

$$\begin{array}{ccc} DE \times_X P & \longrightarrow & P \\ \downarrow & & \downarrow \\ DE & \longrightarrow & X \end{array}$$

Multiplication with the Thom class is given by the pull back diagram

$$\begin{array}{ccc} X \times_{DE} (DE \times_X P) & \longrightarrow & DE \times_X P \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma_0} & DE \end{array}$$

Now observe that

$$X \times_{DE} (DE \times_X P) = \{(x, v, p) \in X \times DE \times P; \sigma_0(x) = v, \pi g(p) = x\}$$

is diffeomorphic to  $P$  by sending  $(x, v, p)$  onto  $p$ .

To finish the proof, one needs to show that the map from the last diagram

$$X \times_{DE} (DE \times_X P) \longrightarrow DE, (x, (v, p)) \mapsto v$$

and the map

$$g : P \longrightarrow DE$$

are cobordant. This is done by a homotopy joining  $g(p)$  and its projection onto the zero section.  $\square$

### 3 Two final remarks

1. The proof of the homological Thom isomorphism follows a similar pattern. Let  $[P, x, g] \in h_q(DE, SE)$ . Consider the pull back diagram

$$\begin{array}{ccc} P \times_{DE} X & \xrightarrow{g'} & X \\ \downarrow & & \downarrow \sigma_0 \\ P & \xrightarrow{g} & DE. \end{array}$$

This gives the cycle  $(P \times_{DE} X, x, g')$  representing a class in  $h_{q-n}(X)$ .

On the other hand, for a cycle  $(M, x, f)$  representing a class in  $h_q(X)$  we get a class in  $h_{q+n}(DE, SE)$  via the pull back

$$\begin{array}{ccc} f^* DE & \longrightarrow & DE \\ \downarrow & & \downarrow \\ M & \longrightarrow & X \end{array}$$

These two constructions are inverse to each other.

Our geometric approach to the homological Thom isomorphism is valid in a more general setting: Let  $E \rightarrow X$  be a vector bundle in the paracompact category. For a class  $[M, x, f]$  of  $h_*(X)$  we can perform the latter pull back, when we replace  $f : M \rightarrow X$  by the composition

$$M \longrightarrow X \longrightarrow BO_n,$$

where the last arrow is the classifying map of the vector bundle.

Taking a smooth universal bundle  $DEO_n \rightarrow BO_n$  we can make the composition  $M \rightarrow BO_n$  smooth. Moreover, the pull back is a smooth, finite dimensional manifold with boundary. In favorable cases it maps to  $(DE, SE)$ .

2. Clearly, the Thom class  $t_E$  can also be viewed as a bivariant class

$$[X, 1_X, \sigma_0] \in h^n(E \xrightarrow{\pi} X).$$

It is not hard to work out the appropriate version of the Thom isomorphism theorem using the intersection product of geometric bivariant theories.

## References

- [1] Baum, P. and Douglas, R.: *K Homology and Index Theory*, Proceedings of Symposia in Pure Mathematics, **38**, Part 1, 117 – 173 (1982)
- [2] Fulton, W. and MacPherson, R.: *Categorical framework for the study of singular spaces*, Memoirs of the AMS, **243**, 1981
- [3] Jakob, M.: *Bivariant theories for smooth manifolds*, Appl. Categ. Struct. **10**, No.3, 279-290 (2002)
- [4] Jakob, M.: *An Alternative Approach to Homology*, Contemp. Math. **265**, 87 – 97 (2000)
- [5] Jakob, M.: *A bordism-type construction of homology*, Manuscr. Math. **96**, 67 – 80 (1998)

M. Jakob  
 Mathematical Institute  
 Univ. of Neuchatel  
 CH-2000 Neuchatel, Switzerland  
 martin.jakob@unine.ch